Solutions to a Couple of IMO-2019 Problems

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I. INTRODUCTION & BACKGROUND

IMO 2019 is ongoing in Bath, United Kingdom. Yesterday (July 16, 2019) was said to be the first day. Some friends posted the three problems of day 1 online, and I took the liberty of working on the first two problems. I did not try the 3rd problem due to lack of interest as well as concern of time I might have to spend.

The following two sections are my solutions to the first two problems. This represents no significance. It is purely for fun remembering old high school days.

II. PROBLEM ONE

A. Problem

Let Z be the set of integers. Determine all functions \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) such that, for all integers \( a \) and \( b \),

\[
f(2a) + 2f(b) = f(f(a + b)).
\]  

(1)

B. Solution

In (1), let \( b = 0, a = x \in \mathbb{Z} \), we get

\[
f(2x) + 2f(0) = f(f(x)),
\]  

(2)

and let \( a = 0, b = x \in \mathbb{Z} \), we get

\[
f(0) + 2f(x) = f(f(x)).
\]  

(3)

Comparing (2) and (3), we get

\[
f(2x) = 2f(x) - f(0).
\]  

(4)

Applying (4) and (3) in (1), we get

\[
2f(a) + 2f(b) - f(0) = f(0) + 2f(a + b),
\]

i.e.

\[
f(a + b) = f(a) + f(b) - f(0).
\]  

(5)

In (5), let \( a = x \in \mathbb{Z}, b = 1 \). Then

\[
f(x + 1) = f(1) - f(0).
\]  

(6)

Hence

\[
f(x) = x + f(0) \quad \text{for some} \; d \in \mathbb{Z}.
\]  

(7)

Applying (7) and (3) leads to

\[
2dx + 3f(0) = d^2x + (d + 1)f(0).
\]

Hence

\[
\begin{cases} 
  d^2 = 2d \\
  (d - 2)f(0) = 0.
\end{cases}
\]  

(8)

From (8), if \( d = 0 \), then \( f(0) = 0 \) and \( f(x) \equiv 0 \); Otherwise, \( d = 2 \). Hence the final solutions are:

\[
f(x) \equiv 0
\]

or

\[
f(x) = 2x + m \quad \text{for any} \; m \in \mathbb{Z}.
\]

III. PROBLEM TWO

A. Problem

In triangle \( ABC \), point \( A_1 \) lies on side \( BC \) and point \( B_1 \) lies on side \( AC \). Let \( P \) and \( Q \) be points on segments \( AA_1 \) and \( BB_1 \), respectively, such that \( PQ \) is parallel to \( AB \). Let \( P_1 \) be a point on line \( PB_1 \), such that \( B_1 \) lies strictly between \( P \) and \( P_1 \), and \( \angle PP_1C = \angle BAC \). Similarly, let \( Q_1 \) be the point on line \( QA_1 \), such that \( A_1 \) lies strictly between \( Q \) and \( Q_1 \), and \( \angle CQ_1Q = \angle CBA \).

Prove that points \( P, Q, P_1 \), and \( Q_1 \) are concyclic.
Fig. 1: Geometric illustration with auxiliary line segments

B. Proof

Extend $PQ$ to intersect $AC$ at $P_2$, and to intersect $BC$ at $Q_2$. Extend $PP_1$ to intersect $BC$ at $G$, and extend $QQ_1$ to intersect $AC$ at $H$. Connect line segments as being shown in Figure 1.

We intend to first prove that $GH$ is parallel to $PQ$ and $AB$. For this, we repeatedly apply Menelaus’s Theorem on different configurations.

Specifically, applying the theorem on $\triangle P_2CQ_2$ and line segment $PG$ leads to

$$\frac{GQ_2}{GC} \times \frac{B_1C}{B_1P_2} \times \frac{PP_2}{PQ_2} = 1 \Rightarrow \frac{GQ_2}{GC} = \frac{B_1P_2}{B_1C} \times \frac{PQ_2}{PP_2},$$

(9)

and applying the theorem on $\triangle P_2CQ_2$ and line segment $QH$ leads to

$$\frac{HP_2}{HC} \times \frac{A_1C}{A_1Q_2} \times \frac{QQ_2}{QP_2} = 1 \Rightarrow \frac{HP_2}{HC} = \frac{A_1Q_2}{A_1C} \times \frac{QP_2}{QQ_2}.$$

(10)

Dividing (10) by (9) leads to

$$\frac{HP_2}{HC} \Big/ \frac{GQ_2}{GC} = \frac{A_1Q_2}{A_1C} \times \frac{PP_2}{PQ_2} \times \frac{B_1C}{B_1P_2} \times \frac{Q{P\_2}}{QQ_2}. $$

(11)

Applying the theorem on $\triangle P_2CQ_2$ and line segment $AA_1$ and $BB_1$ respectively leads to

$$\frac{PP_2}{PQ_2} \times \frac{A_1Q_2}{A_1C} \times \frac{AC}{AP_2} = 1 \Rightarrow \frac{PP_2}{PQ_2} = \frac{A_1Q_2}{A_1C} \times \frac{AP_2}{AC},$$

(12)

$$\frac{B_1C}{B_1P_2} \times \frac{Q{P\_2}}{QQ_2} \times \frac{BC}{BQ_2} = 1 \Rightarrow \frac{B_1C}{B_1P_2} \times \frac{QP_2}{QQ_2} = \frac{BC}{BQ_2}.$$

(13)

Comparing (11), (12) by (13) leads to

$$\frac{HP_2}{HC} \Big/ \frac{GQ_2}{GC} = \frac{AP_2}{AC} \times \frac{BQ_2}{BC} = 1.$$ 

(14)

The last equality was due to the fact that $PQ$ and $P_2Q_2$ are parallel to $AB$. Hence we complete the proof of the statement that $GH$ is parallel to $AB$ and $PQ$. We are now ready to show that the conclusion of the problem holds.

Since $GH$ is parallel to $AB$, we have $\angle CGH = \angle ABC = \angle CQ_1P_1$. Hence points $C, G, H, Q_1$ are co-cyclic.

Similarly, $\angle CHG = \angle BAC = \angle CP_1B_1$. Hence points $C, P_1, G, H$ are co-cyclic.

Note that the two co-cyclic points groups share three common points $C, G, H$. Hence the five points $C, P_1, G, H, Q_1$ are all co-cyclic. Hence

$$\angle CGP_1 = \angle CQ_1P_1.$$

Finally,

$$\angle P_2P_1 = \angle PQ_2G + \angle CGP_1 = \angle ABC + \angle CQ_1P_1 = \angle CQ_1A_1 + \angle CQ_1P_1 = \angle Q{Q\_1}P_1.$$

Hence points $P, P_1, Q_1, Q$ are co-cyclic.